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Lectures on the Error Analysis of Interpolation on Simplicial Triangulations without the Shape-Regularity Assumption Part 2: Lagrange Interpolation on Tetrahedrons

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Abstract
This is the second lecture note on the error analysis of interpolation on simplicial elements without the shape regularity assumption [1]. In this manuscript, we explain the error analysis of Lagrange interpolation on (possibly anisotropic) tetrahedrons. This topic is hardly explained in standard textbooks of the mathematical theory of finite element methods. The authors hope that this manuscript will be merged into a new textbook in future. Therefore, this manuscript is not intended to be a research paper. Supposed readers are students and researchers who are familiar with the mathematical theory of the finite element methods.

Key words: Lagrange interpolation, shape-reguralit y, maximum angle condition, finite elements, tetrahedrons

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§ 1. Lagrange interpolation on tetrahedrons

This is the second lecture note concerning the error analysis of interpolation on simplicial triangulations without the shape regularity assumption. In this note, we will explain the error analysis of Lagrange interpolation on tetrahedrons. To this end, we summarize the results given in [11, 12, 13, 14]. Readers are referred to the first lecture note [15] for the notation, lemmas, and theorems used in this manuscript.

Throughout this paper, $T \subset \mathbb{R}^3$ denotes a tetrahedron with vertices $\mathbf{x}_i$, $i = 1, \cdots, 4$, and all tetrahedrons are assumed to be closed sets. Let λ_i be the barycentric coordinates of a tetrahedron with respect to $\mathbf{x}_i$. By definition, $0 \le \lambda_i \le 1$, $\sum_{i=1}^4 \lambda_i = 1$. Let $\mathbb{N}_0$ be the set of nonnegative integers and $\gamma = (a_1, \cdots, a_4) \in \mathbb{N}_0^4$ be a multi-index. If $|\gamma| := \sum_{i=1}^4 a_i = k$, then $\gamma/k := (a_1/k, \cdots, a_4/k)$ can be regarded as a barycentric coordinate in T. The set $\Sigma^k(T)$ of points on T is defined by

$$\Sigma^k(T) := \left\{ \frac{\gamma}{k} \in T \;\middle|\; |\gamma| = k,\ \gamma \in \mathbb{N}_0^4 \right\}.$$

Let $\mathcal{P}_k(T)$ be the set of polynomials defined on T whose degree is at most k. For a continuous function $v \in C^0(T)$, the Lagrange interpolation $\mathcal{I}_T^k v \in \mathcal{P}_k(T)$ of degree k is defined as

$$v(\mathbf{x}) = (\mathcal{I}_T^k v)(\mathbf{x}), \quad \forall \mathbf{x} \in \Sigma^k(T).$$

Let m, $0 \le m \le k$ be an integer, and p, $1 \le p \le \infty$ be a real. For the mathematical theory of finite element methods, estimating error $|v - \mathcal{I}_T^k v|_{m,p,T}$ of the Lagrange interpolation is an important task. For error analysis, the following condition is usually imposed for the meshes to use in many textbooks [4, 7, 9].

Suppose that $\mathcal{X}$ is a set of (possibly infinitely many) simplicial elements (triangles or tetrahedrons). For $T \in \mathcal{X}$, let $h_T := \mathrm{diam} T$, and ρ_T be the diameter of its inscribed ball.

Assumption 1 (Shape regularity). *The set $\mathcal{X}$ is called* ***shape regular*** *if there exists a constant $\sigma > 0$ such that*

$$\frac{h_T}{\rho_T} \le \sigma, \qquad \forall T \in \mathcal{X}.$$

The shape regularity assumption requires that any element $T \in \mathcal{X}$ is not too "flat", or degenerate. The maximum of the ratio h_T/ρ_T in $\mathcal{X}$ is called its **chunkiness parameter** [4]. The shape regularity condition is sometimes called the **inscribed ball condition**.

Let $\widehat{T}$ be a **reference element**. If we consider about tetrahedrons, the tetrahedron with vertices $(0,0,0)^\top$, $(1,0,0)^\top$, $(0,1,0)^\top$, and $(0,0,1)^\top$ is typically taken as the reference element $\widehat{T}$. Let $\varphi(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ be an affine transformation that maps $\widehat{T}$ to T, where A is a 3×3 regular matrix and $\mathbf{b} \in \mathbb{R}^3$. Error analysis is first performed on the reference element $\widehat{T}$. Then, the pull back $v \circ \varphi$ is used to transfer the result obtained on

$\widehat{T}$ to the "physical element" T. Let $\|A\|$ denote the matrix norm of A associated with the Euclidean norm of $\mathbb{R}^d$ ($d = 2, 3$).

Under the shape regularity assumption, we have the following theorem.

Theorem 2 ([7], Theorem 3.1.4)**.** *Let $\sigma > 0$ be a constant. If $h_T/\rho_T \le \sigma$, then there exists a constant $C = C(\widehat{T}, p, k, m)$ independent of T such that, for $v \in W^{k+1,p}(T)$,*

$$\begin{aligned} |v - \mathcal{I}_T^k v|_{m,p,T} &\le C\|A\|^{k+1}\|A^{-1}\|^m |v|_{k+1,p,T} \\ &\le C\frac{h_T^{k+1}}{\rho_T^m}|v|_{k+1,p,T} \le (C\sigma^m)h_T^{k+1-m}|v|_{k+1,p,T}. \end{aligned}$$

If the chunkiness parameter of $\mathcal{X}$ is not small enough (say, $\sigma > 10$), $\mathcal{X}$ is called **anisotropic**. In numerical simulation, we sometimes need to introduce an *adaptive mesh refinement* technique. In a process of mesh refinements, many anisotropic elements may be generated. With such meshes, the standard theory of finite element methods with the shape regularity assumption cannot be applied. The main purpose of this manuscript is to explain the error analysis of Lagrange interpolation on tetrahedrons *without* the shape regularity assumption.

Figure 1. Two anisotropic triangles; *dagger*: the maximum angle is not close to π, and the circumradius is not large (left), and *brade*: the maximum angle is close to π and the circumradius is large (right).

Let T be a triangle and R_T be its circumradius. Anisotropic triangles can be categorized into only two types as depicted in Figure 1 ([6]). Also, as is explained in [15], the "badness" of an anisotropic triangle can be measured by R_T, and the following theorem is known [15].

Theorem 3 (Circumradius estimates)**.** *Let T be an arbitrary triangle. Then, for the kth-order Lagrange interpolation $\mathcal{I}_T^k$ on T, the estimation*

$$|v - \mathcal{I}_T^k v|_{m,p,T} \le C\left(\frac{R_T}{h_T}\right)^m h_T^{k+1-m}|v|_{k+1,p,T} = CR_T^m h_T^{k+1-2m}|v|_{k+1,p,T}$$

holds for any $v \in W^{k+1,p}(T)$, where the constant $C = C(k, m, p)$ is independent of the geometry of T.

Note that by the laws of sines, we have

$$\frac{R_T}{h_T} = \frac{1}{2\sin\theta_T}, \qquad \frac{\pi}{3} \le \theta_T < \pi$$

where θ_T is the maximum inner angle of T. Hence, if there exists a constant $\theta_{\max} < \pi$ and $\theta_T \le \theta_{\max}$, we have

$$|v - \mathcal{I}_T^k v|_{m,p,T} \le C \left(\frac{R_T}{h_T} \right)^m h_T^{k+1-m} |v|_{k+1,p,T} \le C' h_T^{k+1-m} |v|_{k+1,p,T}.$$

The condition $\theta_T \le \theta_{\max}$ is called the **maximum angle condition** with $\theta_{\max}$ for triangles.

For the case of tetrahedrons, anisotropic tetrahedrons are usually categorized into nine types as depicted in Figure 2 ([6]). Also, as we will see later, the radius of the circumsphere does not represent the "badness" of an anisotropic tetrahedron. These facts suggest that the analysis on anisotropic tetrahedrons is much more complicated than the case of anisotropic triangles.

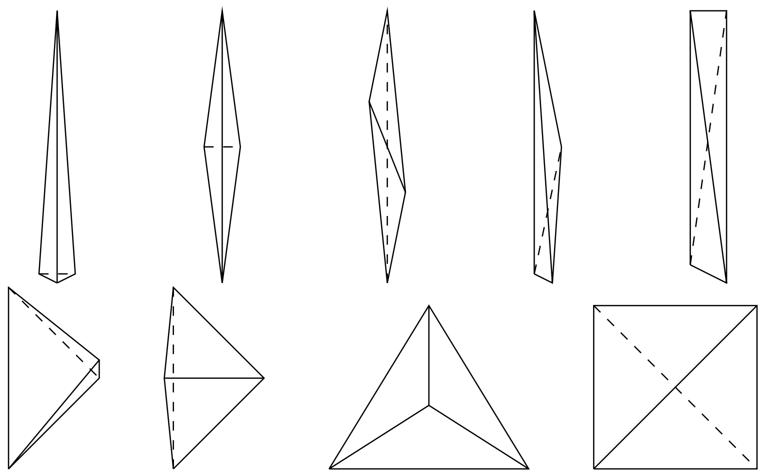


Figure 2. Nine anisotropic tetrahedorns; (top row from left) *spire*, *spear*, *spindle*, *spike*, *splinter*, (bottom row from left) *wedge*, *spade*, *cap*, *sliver*.

Křížek introduced the maximum angle condition for tetrahedrons [16].

Definition 4 (Maximum angle condition for tetrahedrons). *Let $\theta_{\max}$, $\pi/2 \le \theta_{\max} < \pi$ be a constant. Let T be an arbitrary tetrahedron. If all inner angles of the faces of T, and all dihedral angles between two faces of T are less than or equal to $\theta_{\max}$, T is said to satisfy the* ***maximum angle condition*** *with $\theta_{\max}$.*

For the error analysis of Lagrange interpolation on tetrahedrons without the shape regularity condition, the following theorem is known [16, 8].

Theorem 5. *Let $\theta_{\max}$, $\pi/2 \le \theta_{\max} < \pi$ be a constant. Suppose that a tetrahedron T satisfies the maximum angle condition with $\theta_{\max}$. Then, there exists a constant $C = C(\theta_{\max}, p)$ with $p > 2$ such that*

$$|v - \mathcal{I}_T^1 v|_{1,p,T} \le C h_T |v|_{1,p,T},$$

where $C(\theta_{\max}, p) = \mathcal{O}((p-2)^{-1/2})$ as $p \searrow 2$.

By this theorem, we may say that, if a tetrahedron K satisfies the maximum angle condition, the error of the linear Lagrange interpolation is of order $\mathcal{O}(h_K)$ in L^p-norm with $p > 2$.

To extend the above estimation, a theorem similar to Theorem 3 was desired [2]. For that purpose, an immediate idea is to replace the circumradius of a triangle with the radius of circumshpere of a tetrahedron. However, this idea can be immediately rejected by considering the tetrahedron T with vertices $\mathbf{x}_1 := (h, 0, 0)^\top$, $\mathbf{x}_2 := (-h, 0, 0)^\top$, $\mathbf{x}_3 := (0, -h, h^\alpha)^\top$, $\mathbf{x}_4 := (0, h, h^\alpha)^\top$ with $h > 0$ and $\alpha > 0$. This tetrahedron is an example of *sliver* (see Figure 2). Setting $v(x, y, z) := x^2 - h^2 + h^{2-\alpha} z$, we see that $\mathcal{I}_T^1 v \equiv 0$, and a simple computation yields that $|v - \mathcal{I}_T^1 v|_{1,\infty,T} = |v|_{1,\infty,T} \ge h^{2-\alpha}$ and $|v|_{2,\infty,T} = 2$. Hence, if $\alpha > 2$, an inequality such as the one given in Theorem 3 does not hold for the tetrahedron, although the radius of circumshpere of the above T converges to 0 as $h \to 0$.

To express the "badness" of a tetrahedron, the following definition is given [11, 12]. Let h_i $(i = 1, \cdots, 6)$ be the length of edges of T with $h_1 \le \cdots \le h_6 = h_T := \operatorname{diam} T$. Then, we define R_T by

$$R_T := \frac{h_1 h_2 h_T}{|T|} h_T. \tag{1}$$

The following is the main theorem of this manuscript.

[2]Note that Apel [2] presents a different type of error analysis on anisotropic meshes.

Theorem 6 (Main Theorem). *Let T be an arbitrary tetrahedron and R_T be defined by* (1). *Let k and m be integers with $k \geq 1$ and $0 \leq m \leq k$. Let p be taken as*

$$\begin{cases} 2 < p \leq \infty & \text{if } k - m = 0, \\ \frac{3}{2} < p \leq \infty & \text{if } k = 1,\ m = 0, \\ 1 \leq p \leq \infty & \text{if } k \geq 2 \text{ and } k - m \geq 1. \end{cases} \tag{2}$$

For the Lagrange interpolation $\mathcal{I}_T^k v$ of degree k on T, the following estimate holds:

$$B_p^{m,k}(T) := \sup_{u \in \mathcal{T}_p^k(T)} \frac{|u|_{m,p,T}}{|u|_{k+1,p,T}} \leq C_{k,m,p} \left(\frac{R_T}{h_k} \right)^m h_T^{k+1-m},$$

$$|v - \mathcal{I}_T^k v|_{m,p,T} \leq C_{k,m,p} \left(\frac{R_T}{h_T} \right)^m h_T^{k+1-m} |v|_{k+1,p,T}, \quad \forall v \in W^{k+1,p}(T),$$

where $C_{k,m,p}$ is a constant depending on k, m, and p.

Remark. Note that, in (2) and Theorem 6, the restriction $2 < p$ for the case $k = m$ comes from the continuity of the trace operator $\gamma : W^{1,p}(\mathbf{T}) \ni v \mapsto v|_S \in L^1(S)$, where $S \subset \mathbf{T}$ is a non-degenerate segment (see [13, Section 3] and Lemma 19 in Appendix). By the counterexamples given by Shenk [18] and the authors [14], we find that this restriction cannot be improved.

For the maximum angle condition of tetrahedrons, we have the following theorem.

Theorem 7. *Let T be an arbitrary tetrahedron and R_T be defined by* (1). *Then, T satisfies the maximum angle condition with $\theta_{\max} \in [\pi/2, \pi)$, if and only if there exists a fixed constant $D = D(\theta_{\max})$ such that*

$$\frac{R_T}{h_T} \leq D. \tag{3}$$

This theorem implies that, with R_T given in (1), the situation for tetrahedrons is very similar to that of triangles. We immediately obtain the following corollary.

Corollary 8. *Let T be an arbitrary tetrahedron that satisfies the maximum angle condition with $\theta_{\max} \in [\pi/2, \pi)$. Let k and m be integers with $k \geq 1$ and $0 \leq m \leq k$. Let p be taken as* (2). *For the Lagrange interpolation $\mathcal{I}_T^k v$ of degree k on T, the following estimate holds.*

$$|v - \mathcal{I}_T^k v|_{m,p,T} \leq C h_T^{k+1-m} |v|_{k+1,p,T}, \quad \forall v \in W^{k+1,p}(T),$$

where C is a constant depending only on k, m, p, and $\theta_{\max}$.

In the sequel of this lecture note, we will explain the proofs of Theorems 6, 7 in detail.

§ 2. Preliminaries

§ 2.1. Notation

A triangle with vertices $\mathbf{x}_i$ $(i = 1, 2, 3)$ is denoted by $\triangle\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3$. The edge connecting $\mathbf{x}_i$, $\mathbf{x}_j$ and its length are denoted by $\overline{\mathbf{x}_i\mathbf{x}_j}$ and $|\overline{\mathbf{x}_i\mathbf{x}_j}|$, respectively.

§ 2.2. The Sobolev imbedding theorem

Let $1 < p \leq \infty$. From Sobolev's imbedding theorem and Morry's inequality, we have the continuous imbeddings

$$\begin{gathered}
W^{2,p}(T) \subset C^{1,1-3/p}(T), \quad p > 3, \\
W^{2,3}(T) \subset W^{1,q}(T) \subset C^{0,1-3/q}(T), \quad \forall q > 3, \\
W^{2,p}(T) \subset W^{1,3p/(3-p)}(T) \subset C^{0,2-3/p}(T), \quad \frac{3}{2} < p < 3, \\
W^{3,3/2}(T) \subset W^{2,3}(T) \subset W^{1,q}(T) \subset C^{0,1-3/q}(T), \quad \forall q > 3, \\
W^{3,p}(T) \subset W^{2,3p/(3-p)}(T) \subset W^{1,3p/(3-2p)}(T) \subset C^{0,3-3/p}(T), \quad 1 < p < \frac{3}{2}.
\end{gathered}$$

For the imbedding theorem, see [1] and [5]. Although Morry's inequality may not be applied, the continuous imbedding $W^{3,1}(T) \subset C^0(T)$ still holds. For proof of the critical imbedding, see [1, Theorem 4.12] and [4, Lemma 4.3.4]. In the following, we assume that p is taken so that the imbedding $W^{k+1,p}(T) \subset C^0(T)$ holds, that is,

$$1 \leq p \leq \infty, \quad \text{if } k+1 \geq 3 \quad \text{and} \quad \frac{3}{2} < p \leq \infty, \quad \text{if } k+1 = 2.$$

§ 2.3. Classification of tetrahedrons into two types

As noted in [2, 12, 15], to deal with arbitrary tetrahedrons (including anisotropic ones) uniformly, we need to classify tetrahedrons into two types. Let T be an arbitrary tetrahedron. and $\mathbf{x}_i$, $i = 1, \cdots, 4$ be its vertices. Let e_2 be the shortest edge of T and e_1 be the longest edge connected to e_2. We assume that $\mathbf{x}_1$ and $\mathbf{x}_2$ are the endpoints of e_1. Let $\mathbf{x}_3$ be an endpoint of e_2 that is not an endpoint of e_1. Then, e_1 and e_2 are edges of $\triangle\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3$. Note that we still have two cases for assigning $\mathbf{x}_1$ and $\mathbf{x}_2$ as the endpoints of e_1.

Consider the plane that is perpendicular to e_1 and intersects e_1 at its midpoint. Then, $\mathbb{R}^3$ is divided by this plane into two half-spaces. In this situation, we have two cases, and tetrahedrons are classified as either Type 1 or Type 2 accordingly:

- **Case 1**. If one half-space contains three vertices and the other half-space contains one vertex, then T is classified as Type 1.

- **Case 2**. If the two half-spaces contain two vertices each, then T is classified as Type 2.

If the plane contains a vertex, then T is classified as Type 1.

We now introduce the following assignment of the vertices for each case.

- If T is Type 1, the endpoints of e_2 are $\mathbf{x}_1$ and $\mathbf{x}_3$, and the face $\triangle\mathbf{x}_1\mathbf{x}_3\mathbf{x}_4$ belongs to one half-space. Let $\alpha_2 := |\overline{\mathbf{x}_1\mathbf{x}_3}|$.
- If T is Type 2, the endpoints of e_2 are $\mathbf{x}_2$ and $\mathbf{x}_3$, and e_2 and $\overline{\mathbf{x}_1\mathbf{x}_4}$ belongs to the different half-spaces. Let $\alpha_2 := |\overline{\mathbf{x}_2\mathbf{x}_3}|$.

Define $\alpha_1 := |\overline{\mathbf{x}_1\mathbf{x}_2}|$ and $\alpha_3 := |\overline{\mathbf{x}_1\mathbf{x}_4}|$ for both cases.

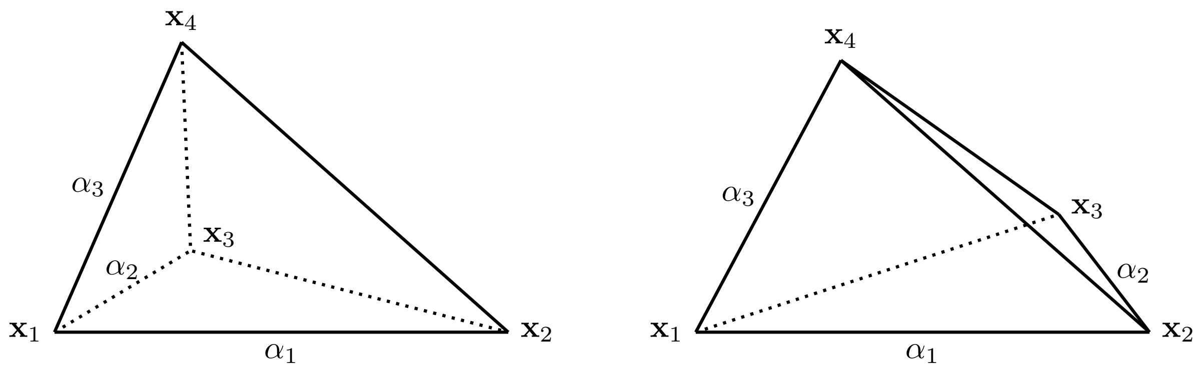


Figure 3. Tetrahedrons of Type 1 (left) and Type 2 (right).

§ 2.4. Standard position of tetrahedrons

For considering the geometry of tetrahedrons, it is convenient to assign coordinates of their vertices explicitly. Suppose that an arbitrary tetrahedron T is taken and classified as explained in Section 2.3. Let the parameters s_1, t_1, s_{21}, s_{22}, t_2 be such that

$$(4)\qquad \begin{cases} s_1^2+t_1^2=1,\ s_1>0,\ t_1>0, \quad \alpha_2 s_1 \le \frac{\alpha_1}{2}, \\ s_{21}^2+s_{22}^2+t_2^2=1,\ t_2>0, \quad \alpha_3 s_{21} \le \frac{\alpha_1}{2}. \end{cases}$$

Suppose that T is Type 1. Then, using translation and rotation, we may move T as $\mathbf{x}_1 \mapsto (0,0,0)^\top$, $\mathbf{x}_2 \mapsto (\alpha_1,0,0)^\top$, and $\mathbf{x}_3 \mapsto (x_3,y_3,0)^\top$ with $y_3>0$. Letting $\theta := \angle\mathbf{x}_2\mathbf{x}_1\mathbf{x}_3$ and $s_1 := \cos\theta$, $t_1 := \sin\theta > 0$, we have $x_3 = \alpha_2 s_1$, $y_3 = \alpha_2 t_1$. Note that, by the assignment of vertices $\mathbf{x}_i$ $(i=1,2,3)$, we have $s_1>0$ (otherwise $|\overline{\mathbf{x}_1\mathbf{x}_2}| < |\overline{\mathbf{x}_3\mathbf{x}_2}|$) and $\alpha_2 s_1 \le \frac{\alpha_1}{2}$. In this situation, $\mathbf{x}_4$ might be below xy-plain (its z-coordiate is negative). If so, we use mirror imaging with respect to xy-plain to make it be above xy-plain (make its z-coordinate positive). Let $(s_{21},s_{22},t_2) := \overrightarrow{\mathbf{x}_1\mathbf{x}_4}/|\overrightarrow{\mathbf{x}_1\mathbf{x}_4}|$. By these procedure, we may assume without loss of generality that T of Type 1 is transformed to a tetrahedron with vertices

$$(5)\quad \mathbf{x}_1=(0,0,0)^\top,\ \mathbf{x}_2=(\alpha_1,0,0)^\top,\ \mathbf{x}_3=(\alpha_2 s_1,\alpha_2 t_1,0)^\top,\ \mathbf{x}_4=(\alpha_3 s_{21},\alpha_3 s_{22},\alpha_3 t_2)^\top.$$

(Recall that $\alpha_2 = |\overline{\mathbf{x}_1\mathbf{a}_3}|$, $\alpha_3 = |\overline{\mathbf{x}_1\mathbf{a}_4}|$, and $\alpha_3 s_{21} \le \alpha_1/2$ by the definition.)

If T is Type 2, we may transform T to a tetrahedron with vertices

(6)
$$\mathbf{x}_1 = (0,0,0)^\top,\ \mathbf{x}_2 = (\alpha_1,0,0)^\top,\ \mathbf{x}_3 = (\alpha_1 - \alpha_2 s_1, \alpha_2 t_1, 0)^\top,\ \mathbf{x}_4 = (\alpha_3 s_{21}, \alpha_3 s_{22}, \alpha_3 t_2)^\top,$$

by a similar manner. We refer to the coordinates in (5), (6) as the **standard position** of T. We always identify T with the tetrahedron with vertices (5), (6). Note that we have

$$|T| = \frac{1}{6}\alpha_1\alpha_2\alpha_3 t_1 t_2, \tag{7}$$

where $|T|$ is the volume of T.

§ 2.5. Reference tetrahedrons

Because we have two types of tetrahedrons, it is convenient to introduce two reference tetrahedrons to deal with them uniformly. Let $\widehat{T}$ and $\widetilde{T}$ be tetrahedrons that have the following vertices (see Figure 4):

$$\widehat{T} \text{ has the vertices } (0,0,0)^\top,\ (1,0,0)^\top,\ (0,1,0)^\top,\ (0,0,1)^\top,$$
$$\widetilde{T} \text{ has the vertices } (0,0,0)^\top,\ (1,0,0)^\top,\ (1,1,0)^\top,\ (0,0,1)^\top.$$

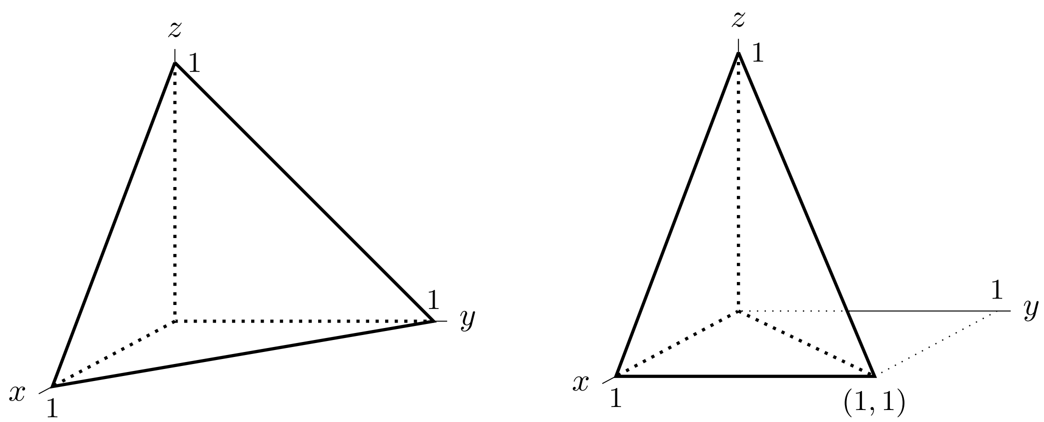


Figure 4. The reference tetrahedrons $\widehat{T}$ (left) and $\widetilde{T}$ (right).

These tetrahedrons are called the **reference tetrahedrons**. In the following, $\widehat{T}$ corresponds to tetrahedrons of Type 1 and $\widetilde{T}$ corresponds tetrahedrons of Type 2. We denote the reference tetrahedrons by $\mathbf{T}$, that is, $\mathbf{T}$ is either of $\{\widehat{T}, \widetilde{T}\}$.

§ 2.6. Linear transformations

For an arbitrary tetrahedron T written as (5) or (6) with parameters (4), we consider an affine transformation from the reference tetrahedrons. Define the matrices $\widehat{A}$,

$\widetilde{A}$, $D_{\alpha_1\alpha_2\alpha_3} \in GL(3,\mathbb{R})$ by

$$\widehat{A} := \begin{pmatrix} 1 & s_1 & s_{21} \\ 0 & t_1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}, \quad \widetilde{A} := \begin{pmatrix} 1 & -s_1 & s_{21} \\ 0 & t_1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}, \quad D_{\alpha_1\alpha_2\alpha_3} := \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}. \tag{8}$$

We immediately confirm that the following lemma holds.

Lemma 9 ([14]). *Let T be an arbitrary tetrahedron in the standard position* (5) *or* (6) *with parameters* (4). *Then, T is transformed from the reference tetrahedron* $\mathbf{T}$ *by $T = \widehat{A}D_{\alpha_1\alpha_2\alpha_3}(\widehat{T})$ for Type 1, or $T = \widetilde{A}D_{\alpha_1\alpha_2\alpha_3}(\widetilde{T})$ for Type 2.*

The linear transformation defined by $D_{\alpha_1\alpha_2\alpha_3}$ is called the **squeezing transformation** [15], and we will show that the squeezing transformation does not reduce approximation property of Lagrange interpolation at all (see Theorem 11).

Note that $\widehat{A}$ and $\widetilde{A}$ are decomposed as $\widehat{A} = X\widehat{Y}$ and $\widetilde{A} = X\widetilde{Y}$ with

$$X := \begin{pmatrix} 1 & 0 & s_{21} \\ 0 & 1 & s_{22} \\ 0 & 0 & t_2 \end{pmatrix}, \qquad \widehat{Y} := \begin{pmatrix} 1 & s_1 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \widetilde{Y} := \begin{pmatrix} 1 & -s_1 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

respectively. We consider the singular values of $\widehat{A}$, $\widetilde{A}$, X, $\widehat{Y}$, and $\widetilde{Y}$. A straightforward computation yields

$$\det\left(X^\top X - \mu I\right) = (1-\mu)\left(\mu^2 - 2\mu + t_2^2\right),$$
$$\det\left(\widehat{Y}^\top \widehat{Y} - \mu I\right) = \det\left(\widetilde{Y}^\top \widetilde{Y} - \mu I\right) = (1-\mu)\left(\mu^2 - 2\mu + t_1^2\right).$$

Thus, we find that, setting $\mathbf{s}_1 := |s_1|$ and $\mathbf{s}_2 := (s_{21}^2 + s_{22}^2)^{1/2}$,

$$\|X\| = (1+\mathbf{s}_2)^{1/2}, \quad \|X^{-1}\| = (1-\mathbf{s}_2)^{-1/2},$$
$$\|Y\| = (1+\mathbf{s}_1)^{1/2}, \quad \|Y^{-1}\| = (1-\mathbf{s}_1)^{-1/2}, \qquad Y = \widehat{Y} \text{ or } Y = \widetilde{Y},$$
$$\|A\| \le \prod_{i=1}^{2}(1+\mathbf{s}_i)^{1/2}, \quad \|A^{-1}\| \le \prod_{i=1}^{2}(1-\mathbf{s}_i)^{-1/2}, \qquad A = \widehat{A} \text{ or } A = \widetilde{A}. \tag{9}$$

Note that

$$\mathbf{s}_i^2 + t_i^2 = 1,\ i = 1,2 \quad \text{and} \quad \|A^{-1}\| \le \prod_{i=1}^{2}(1-\mathbf{s}_i)^{-1/2} = \prod_{i=1}^{2}\frac{(1+\mathbf{s}_i)^{1/2}}{t_i}. \tag{10}$$

§ 2.7. Another geometric quantities of tetrahedrons

In (1), a quantity R_T is defined for a tetrahedron T. Here, we define another quantity H_T [12], which represent the geometry of T, by

$$H_T := \frac{\alpha_1\alpha_2\alpha_3}{|T|}h_T = \frac{6h_T}{t_1t_2},$$

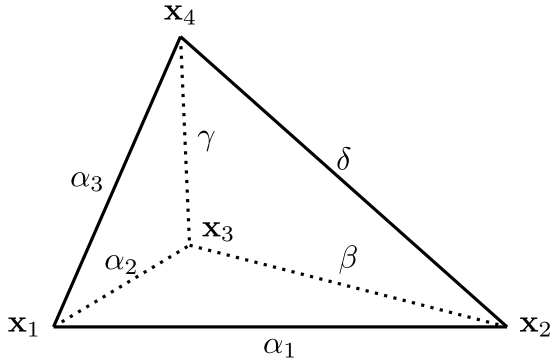
where the last equation is from (7). Then, the following lemma holds [12, Lemma 3].

Lemma 10. *The two quantities R_T and H_T are equivalent. That is, for an arbitrary tetrahedron T, we have*

$$\frac{1}{2}H_T \le R_T \le 2H_T. \tag{11}$$

Proof. Suppose that we have a triangle with the edge lengths $h_1 \le h_2 \le h_3$. Then, $\frac{1}{2}h_3 < h_2 \le h_3$. Let T be an arbitrary tetrahedron T in the standard position.

Case 1. Suppose that T is of Type 1. Set $\beta := |\overline{\mathbf{x}_2\mathbf{x}_3}|$, $\gamma := |\overline{\mathbf{x}_3\mathbf{x}_4}|$, and $\delta := |\overline{\mathbf{x}_2\mathbf{x}_4}|$.



By the definition of the standard position, we have

$$\alpha_2 \le \min\{\alpha_3, \beta, \gamma\} \le \max\{\alpha_3, \beta, \gamma\} \le \alpha_1.$$

Hence, we have either $h_T = \alpha_1$ or $h_T = \delta$. Note that $\overline{\mathbf{x}_1}$ the shortest edge of the triangle $\triangle \mathbf{x}_1\mathbf{x}_2\mathbf{x}_4$ because $\mathbf{x}_1$ an belong to the same half-space.

Hence, we have $\alpha_3 \le \delta$ and

$$\alpha_1 \le h_T < 2\alpha_1, \quad \text{or} \quad \frac{1}{2}h_T < \alpha_1 \le h_T.$$

So far, we realize that either $h_2 = \alpha_3$, $h_2 = \beta$, or $h_2 = \gamma$. Recall that $\alpha_2 = h_1$. In the following, we check each case.

- Case of $h_2 = \alpha_3$. In this case, we have $\alpha_1\alpha_2\alpha_3 = \alpha_1 h_1 h_2$, and

$$\alpha_1\alpha_2\alpha_3 \le h_1 h_2 h_T < 2\alpha_1\alpha_2\alpha_3 \quad \text{amd} \quad H_T \le R_T < 2H_T.$$

- Case of $h_2 = \beta$. Note that $h_2 = \beta \le \alpha_3$, and $\overline{\mathbf{x}_1\mathbf{x}_2}$ and $\overline{\mathbf{x}_1\mathbf{x}_3}$ are the longest and shortest edges of $\triangle \mathbf{x}_1\mathbf{x}_2\mathbf{x}_3$, respectively. Therefore, we have

$$\frac{1}{2}\alpha_3 \le \frac{1}{2}\alpha_1 < \beta = h_2 \le \alpha_3 \le \alpha_1.$$

  This means that

$$\frac{1}{2}\alpha_1\alpha_2\alpha_3 < h_1 h_2 h_T \le 2\alpha_1\alpha_2\alpha_3 \quad \text{and} \quad \frac{1}{2}H_T < R_T \le 2H_T.$$

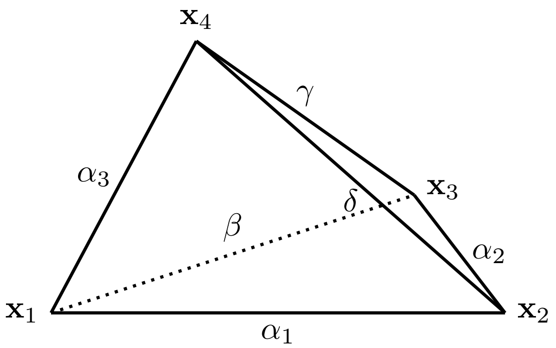
- Case of $h_2 = \gamma$. Note that $h_2 = \gamma \le \alpha_3$, and $\overline{\mathbf{x}_1\mathbf{x}_4}$ and $\overline{\mathbf{x}_1\mathbf{x}_3}$ are the longest and shortest edges of $\triangle \mathbf{x}_1\mathbf{x}_3\mathbf{x}_4$, respectively. Therefore, we have

$$\frac{1}{2}\alpha_3 < \gamma = h_2 \le \alpha_3.$$

  This implies

$$\frac{1}{2}\alpha_1\alpha_2\alpha_3 < h_1 h_2 h_T \le 2\alpha_1\alpha_2\alpha_3 \quad \text{and} \quad \frac{1}{2}H_T < R_T \le 2H_T.$$

Case 2. Suppose that T is of Type 2. Set $\beta := |\mathbf{x}_1\mathbf{x}_3|$, $\gamma := |\mathbf{x}_3\mathbf{x}_4|$, and $\delta := |\mathbf{x}_2\mathbf{x}_4|$.



By the definition of the standard position, we have

$$\alpha_2 \le \min\{\beta, \gamma, \delta\} \le \max\{\beta, \gamma, \delta\} \le \alpha_1.$$

Note that $\overline{\mathbf{x}_1\mathbf{x}_2}$ is the longest edge of the triangle $\triangle \mathbf{x}_1$ because $\mathbf{x}_1$ and $\mathbf{x}_4$ belong to the same half-space. Henc have $\alpha_3 \le \delta \le \alpha_1 = h_T$.

Therefore, we realize that either $h_2 = \alpha_3$, $h_2 = \beta$, or $h_2 = \gamma$. In the following, we check each case.

- Case of $h_2 = \alpha_3$. In this case, we have $\alpha_1\alpha_2\alpha_3 = h_1 h_2 h_T$ and $H_T = R_T$.
- Case of $h_2 = \beta$. Note that $h_2 = \beta \le \alpha_3$, and $\overline{\mathbf{x}_1\mathbf{x}_2}$ and $\overline{\mathbf{x}_2\mathbf{x}_3}$ are the longest and shortest edges of $\triangle \mathbf{x}_1\mathbf{x}_2\mathbf{x}_3$, respectively. Therefore, we have

$$\frac{1}{2}\alpha_3 \le \frac{1}{2}\alpha_1 < \beta = h_2 \le \alpha_3 \le \alpha_1.$$

  This implies

$$\frac{1}{2}\alpha_1\alpha_2\alpha_3 < h_1 h_2 h_T \le \alpha_1\alpha_2\alpha_3 \quad \text{and} \quad \frac{1}{2}H_T < R_T \le H_T.$$

- Case of $h_2 = \gamma$. Note that $h_2 = \gamma \le \alpha_3 \le \delta$, and $\overline{\mathbf{x}_2\mathbf{x}_4}$ and $\overline{\mathbf{x}_1\mathbf{x}_3}$ are the longest and shortest edges of $\triangle \mathbf{x}_2\mathbf{x}_3\mathbf{x}_4$, respectively. Therefore, we have

$$\frac{1}{2}\alpha_3 \le \frac{1}{2}\delta < \gamma = h_2 \le \alpha_3 \le \delta.$$

  This implies

$$\frac{1}{2}\alpha_1\alpha_2\alpha_3 < h_1 h_2 h_T \le \alpha_1\alpha_2\alpha_3 \quad \text{and} \quad \frac{1}{2}H_T < R_T \le H_T.$$

Therefore, all cases are checked and the proof is completed. □

Remark. In [14], the *projected circumradius* $\widetilde{R}_T$ is defined for a tetrahedron T as follows. Take any facet B of T, and suppose that T is transformed by translation and rotation so that B is on xy-plain. Let P_{xz} be the perpendicular projection of $\mathbb{R}^3$ onto xz-plain; $P_{xz}(x, y, z) := (x, 0, z)$. Note that the image $P_{xy}(T)$ is a triangle, and let R_0 be its circumradius. Now, consider rotating T around the circumcenter of B on xy-plain. Let T_θ be the rotated tetrahedron, where θ is the angle of the rotation. Let R_θ be the circumradius of $P_{xz}(T_\theta)$ (see Figure 5). Then, define

$$R_P := \max_{\theta \in [-\pi/2, \pi/2]} R_\theta, \qquad \widetilde{R}_T := \min_B \frac{R_P R_B}{h_B},$$

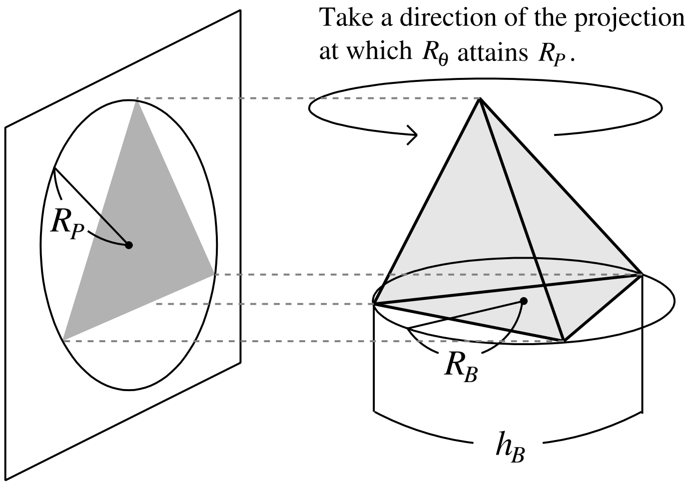


Figure 5. The image of the projected circumradius of T.

where R_B is the circumradius of B, $h_B := \mathrm{diam} B$, and the minimum is taken over all the facets of T. In [14], a theorem similar to Theorem 6 is proved using $\widetilde{R}_T$. It is conjectured that R_T defined by (1) and the projected circumradius $\widetilde{R}_T$ are equivalent.

While the circumradius of a triangle is a good and simple geometric quantity that represent its "badness" (or "goodness"), it is not so clear what is the *best* geometric quantity of a tetrahedron that represents its "badness".

§ 2.8. Squeezing theorem

As is explained in Section 2.4, we may assume without loss of generality that an arbitrary tetrahedron T may be in the standard position. Let $T_{\alpha_1\alpha_2\alpha_3} := D\mathbf{T}$, where the diagonal matrix D is defined in (8). We define the set $\mathcal{T}_p^k(T) \subset W^{k+1,p}(T)$ by

$$\mathcal{T}_p^k(T) := \left\{ v \in W^{k+1,p}(T) \;\middle|\; v(\mathbf{x}) = 0,\ \forall \mathbf{x} \in \Sigma^k(T) \right\}.$$

Then, we have the following **squeezing theorem**.

Theorem 11. *Let k and m be integers with $k \ge 1$ and $0 \le m \le k$. Let p be taken as (2). Then, there exists a constant $C_{k,m,p}$ depending on k, m, p, but independent of α_i $(i = 1, 2, 3)$ such that*

$$B_p^{m,k}(T_{\alpha_1\alpha_2\alpha_3}) := \sup_{v \in \mathcal{T}_p^k(T_{\alpha_1\alpha_2\alpha_3})} \frac{|v|_{m,p,T_{\alpha_1\alpha_2\alpha_3}}}{|v|_{k+1,p,T_{\alpha_1\alpha_2\alpha_3}}} \le \left(\max_{i=1,2,3} \alpha_i \right)^{k+1-m} C_{k,m,p}.$$

Proof. Because the proof is very similar to that of [15, Theorem 21], we give it in Appendix. □

§ 3. Proof of Theorem 6

In this section, we prove Theorem 6 using the setting prepared so far. Suppose that an arbitrary tetrahedron T is in the standard position. Recall that $T = AD(\mathbf{T})$ and $T_{\alpha_1\alpha_2\alpha_3} := D\mathbf{T}$, where $(A, \mathbf{T}) = (\widehat{A}, \widehat{T})$ or $(A, \mathbf{T}) = (\widetilde{A}, \widetilde{T})$ defined by (8) according to the type of T. Let $v \in W^{k+1,p}(T)$, and $\tilde{v} \in W^{k+1,m}(T_{\alpha_1\alpha_2\alpha_3})$ be defined by $\tilde{v}(x) = v(A\mathbf{x})$. Then, it follows from [15, Lemma 12] that

$$|v|_{m,p,T} \le 3^{m\mu(p)} t^{1/p} \|A^{-1}\|^m |\tilde{v}|_{m,p,T_{\alpha_1\alpha_2\alpha_3}},$$
$$3^{-(k+1)\mu(p)} t^{1/p} \|A\|^{-(k+1)} |\tilde{v}|_{k+1,p,T_{\alpha_1\alpha_2\alpha_3}} \le |v|_{k+1,p,T}.$$

Combining the above inequalities and Theorem 11, we obtain

$$\begin{aligned}
\frac{|v|_{m,p,T}}{|v|_{k+1,p,T}} &\le c_{k,m,p} \|A\|^{k+1} \|A^{-1}\|^m \frac{|\tilde{v}|_{m,p,T_{\alpha_1\alpha_2\alpha_3}}}{|\tilde{v}|_{k+1,p,T_{\alpha_1\alpha_2\alpha_3}}} \\
&\le c_{k,m,p} C_{k,m,p} \|A\|^{k+1} \|A^{-1}\|^m \left(\max_{i=1,2,3} \alpha_i \right)^{k+1-m} \\
&\le c_{k,m,p} C_{k,m,p} \|A\|^{k+1} \|A^{-1}\|^m h_T^{k+1-m}.
\end{aligned}$$

where $c_{k,m,p} := 3^{(k+1+m)\mu(p)}$. Therefore, we obtain the following lemma.

Lemma 12. *For an arbitrary triangle T in the standard position, we have*

$$B_p^{m,k+1}(T) := \sup_{v \in \mathcal{T}_p^k(T)} \frac{|v|_{m,p,T}}{|v|_{k+1,p,T}} \le c_{k,m,p} C_{k,m.p} \|A\|^{k+1} \|A^{-1}\|^m h_T^{k+1-m}.$$

Therefore, inserting $v - \mathcal{I}_T^k v \in \mathcal{T}_p^k(T)$ into v, we have

$$|v - \mathcal{I}_T^k v|_{m,p,T} \le c_{k,m,p} C_{k,m.p} \|A\|^{k+1} \|A^{-1}\|^m h_T^{k+1-m} |v|_{k+1,p,T}, \quad \forall v \in W^{k+1,p}(T).$$

We attempt to obtain upper bounds of $\|A\|$ and $\|A^{-1}\|$. From (9), (10), (1), and (11), we know that

$$\|A\| \le 2, \qquad \|A^{-1}\| \le \frac{2}{t_1 t_2} = \frac{H_T}{3h_T} \le \frac{2R_T}{3h_T}.$$

Hence, redefining the constant $C_{k,m,p}$ (recall that the Sobolev (semi-)norms may be affected by rotation up to a constant [15, (16)]), Theorem 6 is proved.

§ 4. Proof of Theorem 7

In this section, we give a proof of Theorem 7. For the proof, we introduce the following notation convention on T. Let F_i be the face of T opposite to $\mathbf{x}_i$. We denote the dihedral angle between the faces F_i and F_j by $\psi^{i,j}$. Note that $\psi^{i,j} = \psi^{j,i}$. Furthermore, we denote the internal angle at $\mathbf{x}_j$ on F_i by θ^i_j, and the angle between F_i and $\overline{\mathbf{x}_i\mathbf{x}_j}$ by ϕ^i_j.

Table 1. Notation convention on T $(i, j = 1,2,3,4,\ i \neq j)$.

$\mathbf{x}_i$	the vertices of T.
F_i	the face opposite to $\mathbf{x}_i$.
$\psi^{i,j}$	the dihedral angle between F_i and F_j.
θ^i_j	the internal angle of F_i at $\mathbf{x}_j$.
ϕ^i_j	the angle between F_i and $\overline{\mathbf{x}_i\mathbf{x}_j}$.

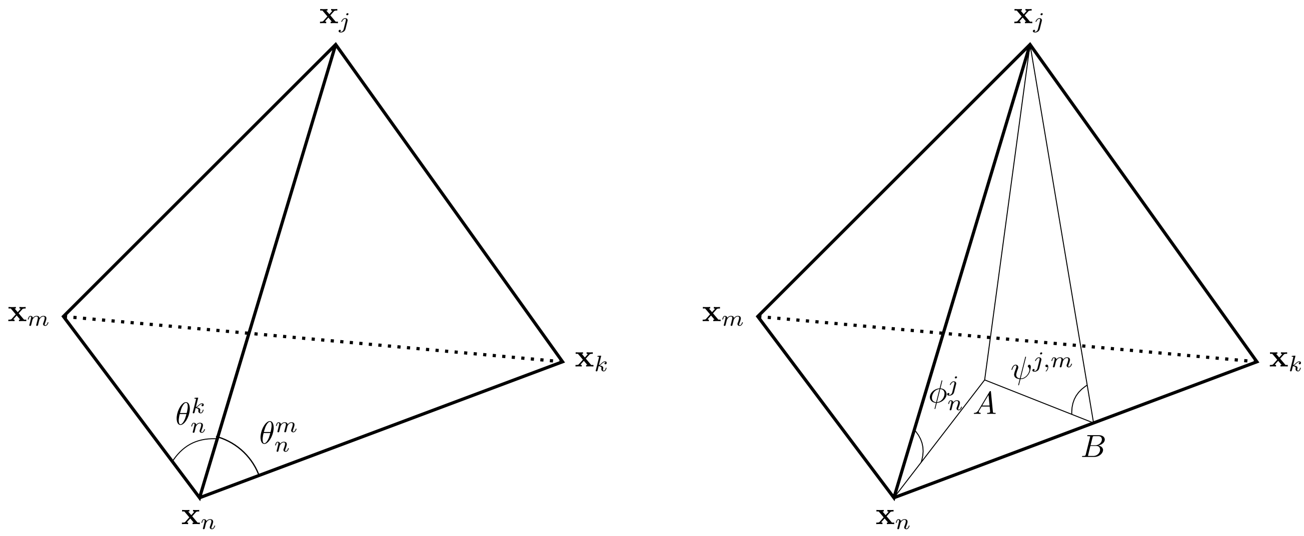


Figure 6. Definitions of the angles on T.

Let A and B be the feet of perpendicular lines from $\mathbf{x}_j$ to F_j and from $\mathbf{x}_j$ to $\overline{\mathbf{x}_n\mathbf{x}_k}$, respectively (see Figure 6). Then, we have

$$|\overline{\mathbf{x}_j\mathbf{x}_n}| \sin\phi^j_n = |\overline{\mathbf{x}_j A}| = |\overline{\mathbf{x}_j B}| \sin\psi^{j,m} = |\overline{\mathbf{x}_j\mathbf{x}_n}| \sin\theta^m_n \sin\psi^{j,m}.$$

A similar equation holds for ϕ^j_n, θ^k_n, and $\psi^{k,j}$. Therefore,

$$\begin{aligned} \sin\phi^j_n = \sin\theta^k_n \sin\psi^{k,j} = \sin\theta^m_n \sin\psi^{m,j} \\ j = 1,2,3,4, \quad m,n,k \in \{1,2,3,4\}\backslash\{j\}. \end{aligned} \tag{12}$$

In the following, we abbreviate "maximum angle condition" as MAC.

Lemma 13 (Cosine rules on tetrahedrons)**.** *Let $T \subset \mathbb{R}^3$ be a tetrahedron. Let $j = 1,2,3,4$ and $\{k,m,n\} = \{1,2,3,4\}\backslash\{j\}$. Then, we have*

$$\begin{aligned} \cos\theta^k_j &= \cos\theta^m_j \cos\theta^n_j + \sin\theta^m_j \sin\theta^n_j \cos\psi^{m,n}, \\ \cos\psi^{n,m} &= \sin\psi^{m,k} \sin\psi^{n,k} \cos\theta^k_j - \cos\psi^{m,k} \cos\psi^{n,k}. \end{aligned} \tag{13}$$

Proof. See [10, 19]. □

Lemma 14. *Let $T \subset \mathbb{R}^2$ be a triangle and let θ_i $(i = 1, 2, 3)$ be the internal angles of T with $\theta_1 \le \theta_2 \le \theta_3$. If there exists $\theta_{\max} \in [\pi/3, \pi)$ such that $\theta_3 \le \theta_{\max}$, then we have*

$$\sin\theta_2,\ \sin\theta_3 \ge \min\left\{ \sin\frac{\pi - \theta_{\max}}{2}, \sin\theta_{\max} \right\}. \tag{14}$$

Proof. Because $\theta_1 + \theta_2 + \theta_3 = \pi$, the assumptions yield

$$2\theta_2 \ge \theta_1 + \theta_2 = \pi - \theta_3 \ge \pi - \theta_{\max} \quad \text{and} \quad \frac{\pi - \theta_{\max}}{2} \le \theta_2 \le \theta_3 \le \theta_{\max},$$

which implies (14). □

Lemma 15. *For $\gamma \in [\pi/3, \pi)$, we have*

$$0 < \frac{\cos\gamma + 1}{\sin\frac{\gamma}{2} + 1} \le 1.$$

Proof. This lemma can be proved immediately from

$$\frac{\cos\gamma + 1}{\sin\frac{\gamma}{2} + 1} = 2\left(1 - \sin\frac{\gamma}{2}\right), \qquad \frac{\pi}{6} \le \frac{\gamma}{2} < \frac{\pi}{2}, \qquad \frac{1}{2} \le \sin\frac{\gamma}{2} < 1. \qquad \square$$

Lemma 16. *Let $T \subset \mathbb{R}^3$ be a tetrahedron. Suppose that T satisfies the MAC with $\theta_{\max} \in [\pi/3, \pi)$. Additionally, assume that θ_n^j is not the minimum angle of face $F_j = \triangle P_m P_n P_k$, and $\theta_n^j < \pi/2$, where $j = 1, 2, 3, 4$ and $\{m, n, k\} = \{1, 2, 3, 4\}\backslash\{j\}$. Then, setting δ to*

$$\sin\delta = \left(\frac{\cos\theta_{\max} + 1}{\sin\frac{\theta_{\max}}{2} + 1} \right)^{1/2}, \qquad 0 < \delta \le \frac{\pi}{2},$$

we have either

$$\psi^{m,j} \ge \delta, \quad \text{or} \quad \psi^{k,j} \ge \delta.$$

Proof. From Lemma 15, we have

$$0 < \frac{\cos\theta_{\max} + 1}{\sin\frac{\theta_{\max}}{2} + 1} \le 1,$$

and we confirm that δ is well-defined.

The proof is by contradiction. Suppose that

$$0 < \psi^{m,j} < \delta \quad \text{and} \quad 0 < \psi^{k,j} < \delta.$$

Then, we have $0 < \sin\psi^{m,j} \sin\psi^{k,j} < \sin^2\delta$ and $1 > \cos\psi^{m,j}\cos\psi^{k,j} > \cos^2\delta$. From Lemma 14 and the assumption, we have

$$\frac{\pi - \theta_{\max}}{2} \le \theta_n^j < \frac{\pi}{2}, \qquad 0 < \cos\theta_n^j \le \cos\left(\frac{\pi - \theta_{\max}}{2}\right) = \sin\frac{\theta_{\max}}{2}.$$

Thus, we obtain

$$\sin\psi^{m,j}\sin\psi^{k,j}\cos\theta_n^j < \sin^2\delta\sin\frac{\theta_{\max}}{2}.$$

The cosine rule (13) and the above inequalities yield

$$\begin{aligned}\cos\psi^{m,k} &= \sin\psi^{m,j}\sin\psi^{k,j}\cos\theta_n^j - \cos\psi^{m,j}\cos\psi^{k,j} \\ &< \sin^2\delta\sin\frac{\theta_{\max}}{2} - (1-\sin^2\delta) \\ &= \frac{\cos\theta_{\max}+1}{\sin\frac{\theta_{\max}}{2}+1}\left(\sin\frac{\theta_{\max}}{2}+1\right) - 1 = \cos\theta_{\max},\end{aligned}$$

which contradicts the MAC: $\psi^{m,k} \le \theta_{\max}$. □

Corollary 17. *Under the assumptions of Lemma 16, we have*

$$\sin\psi^{m,j} \ge C_0, \quad \text{or} \quad \sin\psi^{k,j} \ge C_0, \qquad C_0 := \min\{\sin\delta, \sin\theta_{\max}\}.$$

Lemma 18. *For $j = 1,2,3,4$, let $\{m,n,k\} = \{1,2,3,4\}\backslash\{j\}$. Let $p \in \{m,n,k\}$, and $\{q,r\} = \{m,n,k\}\backslash\{p\}$. Suppose that there exists a positive constant M with $0 < M < 1$ such that $\sin\phi_p^j \sin\theta_n^j \ge M$. Then, setting $\gamma(M) := \pi - \sin^{-1} M$ ($\frac{\pi}{2} < \gamma(M) < \pi$), the MAC with $\gamma(M)$ is satisfied on faces F_j, F_q, F_r, and $\psi^{j,q}$, $\psi^{j,r} \le \gamma(M)$.*

Proof. From the assumption, we have

$$M \le \sin\phi_p^j \sin\theta_n^j \le \sin\theta_n^j \qquad \text{and} \qquad M \le \sin\phi_p^j.$$

Hence, the definition of $\gamma(M)$ yields $\pi - \gamma(M) \le \theta_n^j \le \gamma(M)$. Because $\theta_n^j + \theta_m^j + \theta_k^j = \pi$, we see that θ_m^j, $\theta_k^j < \theta_m^j + \theta_k^j \le \gamma(M)$. That is, the MAC with $\gamma(M)$ is satisfied on face $F_j = \triangle P_m P_n P_k$.

Moreover, it follows from (12) that

$$\begin{aligned}M &\le \sin\phi_p^j = \sin\theta_p^q \sin\psi^{q,j} = \sin\theta_p^r \sin\psi^{r,j} \\ &\le \sin\theta_p^q,\ \sin\theta_p^r,\ \sin\psi^{r,j},\ \sin\psi^{q,j}\end{aligned}$$

By the same reasoning, we find that the MAC with $\gamma(M)$ is satisfied on faces F_q and F_r, and $\psi^{j,q}$, $\psi^{j,r} \le \gamma(M)$. □

In the following, we prove Theorem 7 using H_T instead of R_T. We divide the proof into four cases.

§ 4.1. Type 1: Proof of "MAC implies (3)"

First, we suppose that T is of Type 1 and satisfies the MAC with $\theta_{\max}$, $\pi/3 \le \theta_{\max} < \pi$. Because $|T| = \frac{1}{6}\alpha_1\alpha_2\alpha_3\sin\theta_1^4\sin\phi_1^4$, we have

$$\frac{H_T}{h_T} = \frac{\alpha_1\alpha_2\alpha_3}{|T|} = \frac{6}{\sin\theta_1^4\sin\phi_1^4}.$$

From the definition of Type 1, we realize that $\theta_2^4 \le \theta_1^4 \le \theta_3^4$, that is, θ_3^4 and θ_2^4 are the maximum and minimum angles of face $F_4 = \triangle P_1P_2P_3$, respectively. Thus, it follows from Lemma 14 that

$$\frac{\pi - \theta_{\max}}{2} \le \theta_1^4 \le \theta_{\max}, \quad \sin\theta_1^4 \ge \min\left\{\sin\frac{\pi-\theta_{\max}}{2}, \sin\theta_{\max}\right\} =: C_1.$$

Additionally, we may apply Lemma 16 to θ_1^4 and F_4, and find that either $\psi^{2,4} \ge \delta$ or $\psi^{3,4} \ge \delta$, where $\delta = \delta(\theta_{\max})$, $0 < \delta \le \pi/2$ is defined as

$$\sin\delta = \left(\frac{\cos\theta_{\max}+1}{\sin\frac{\theta_{\max}}{2}+1}\right)^{1/2}. \tag{15}$$

Suppose that $\psi^{2,4} \ge \delta$. By Corollary 17 and (12), we have

$$\sin\phi_1^4 = \sin\theta_1^2 \sin\psi^{2,4} \ge C_0 \sin\theta_1^2,$$

where C_0 is the constant defined in Corollary 17. By the definition of Type 1, θ_1^2 is not the minimum angle of $F_2 = \triangle P_1P_3P_4$, and therefore, we have

$$\frac{\pi - \theta_{\max}}{2} \le \theta_1^2 \le \theta_{\max}, \quad \sin\theta_1^2 \ge C_1.$$

Thus, we obtain $\sin\phi_1^4 \ge C_0C_1$.

Next, suppose that $\psi^{3,4} \ge \delta$. Replacing $\psi^{2,4}$, θ_1^2, and F_2 with $\psi^{3,4}$, θ_1^3, and F_3 in the above argument, we obtain $\sin\phi_1^4 \ge C_0C_1$ in the same manner.

Gathering the above results, we conclude that

$$\frac{H_T}{h_T} = \frac{6}{\sin\theta_1^4 \sin\phi_1^4} \le \frac{6}{C_0C_1^2} =: D$$

in both cases, that is, (3) holds.

§ 4.2. Type 1: Proof of "(3) implies MAC"

Now, we suppose that T is of Type 1 and

$$\frac{H_T}{h_T} = \frac{\alpha_1\alpha_2\alpha_3}{|T|} = \frac{6}{\sin\theta_1^4 \sin\phi_1^4} \le D.$$

Because $\theta_1^4 < \pi/2$ and $\sin\theta_1^4 \sin\phi_1^4 < 1$, we have

$$\sin\theta_1^4 \sin\phi_1^4 \ge \frac{6}{D} =: M, \qquad 0 < M < 1.$$

By Lemma 18 with $j = 4$ and $p = 1$, setting $\gamma(M) := \pi - \sin^{-1} M$, we have $\frac{\pi}{2} < \gamma(M) < \pi$, and the MAC with $\gamma(M)$ is satisfied on F_2, F_3, F_4, and $\psi^{2,4}$, $\psi^{3,4} \le \gamma(M)$.

Note that $|T| = \frac{1}{6}\alpha_1\alpha_2\alpha_3 \sin\theta_1^3 \sin\phi_1^3$, and we have

$$\frac{H_T}{h_T} = \frac{\alpha_1\alpha_2\alpha_3}{|T|} = \frac{6}{\sin\theta_1^3 \sin\phi_1^3} \le D.$$

Thus, by Lemma 18 with $j = 3$ and $p = 1$, we find that $\psi^{2,3} \le \gamma(M)$.

Because $|\overline{P_3P_4}| < |\overline{P_1P_4}| + |\overline{P_1P_3}| \le 2\alpha_3$ on $F_2 = \triangle P_1P_3P_4$ and $|\overline{P_2P_3}| \le \alpha_1$, we note that

$$|T| = \frac{1}{6}\alpha_2|\overline{P_2P_3}||\overline{P_3P_4}| \sin\theta_3^1 \sin\phi_3^1 < \frac{1}{3}\alpha_1\alpha_2\alpha_3 \sin\theta_3^1 \sin\phi_3^1.$$

Thus, we have

$$D \ge \frac{H_K}{h_K} > \frac{3}{\sin\theta_3^1 \sin\phi_3^1} \quad \text{and} \quad \sin\theta_3^1 \sin\phi_3^1 > \frac{3}{D} = \frac{M}{2}.$$

From Lemma 18, setting $\gamma(M/2) := \pi - \sin^{-1}(M/2)$, we have $\frac{\pi}{2} < \gamma(M/2) < \pi$ and MAC with $\gamma(M/2)$ is satisfied on F_1, and $\psi^{2,1}$, $\psi^{4,1} \le \gamma(M/2)$.

The final thing to prove is the MAC for $\psi^{1,3}$. From the cosine rule (13), we have

$$\cos\psi^{1,3} = \sin\psi^{3,4} \sin\psi^{4,1} \cos\theta_2^4 - \cos\psi^{3,4}\cos\psi^{4,1}.$$

By the definition of Type 1, the angle θ_2^4 is the minimum angle of $F_4 = \triangle P_1P_2P_3$, and therefore, we have

$$\cos\theta_2^4 \ge \frac{1}{2},\ \sin\psi^{3,4}\sin\psi^{4,1}\cos\theta_2^4 > 0, \quad \text{and } \cos\psi^{1,3} > -\cos\psi^{3,4}\cos\psi^{4,1}.$$

From the above argument, we have $\sin\psi^{3,4} > M$, $\sin\psi^{4,1} > M/2$, and

$$\begin{aligned}\cos\psi^{1,3} &> -\cos\psi^{3,4}\cos\psi^{4,1} \ge -|\cos\psi^{3,4}||\cos\psi^{4,1}| \\ &= -\sqrt{1-\sin^2\psi^{3,4}}\sqrt{1-\sin^2\psi^{4,1}} > -\sqrt{1-M^2}\sqrt{1-\frac{M^2}{4}} > -1.\end{aligned}$$

Therefore, we conclude that

$$\psi^{1,3} < \cos^{-1}\left(-\sqrt{1-M^2}\sqrt{1-\frac{M^2}{4}}\right) < \pi,$$

and T satisfies the MAC with

$$\theta_{\max} := \max\left\{\gamma(M/2), \cos^{-1}\left(-\sqrt{1-M^2}\sqrt{1-\frac{M^2}{4}}\right)\right\}.$$

§ 4.3. Type 2: Proof of "MAC implies (3)"

First, we suppose that T is of Type 2 and satisfies the MAC with $\theta_{\max} \in [\pi/3, \pi)$. The proof is very similar to that described in Section 4.1.

By the definition of Type 2, $\alpha_3 = |\overline{P_1P_4}| \le |\overline{P_2P_4}|$. Because

$$|T| = \frac{1}{6}\alpha_1\alpha_2\alpha_3 \sin\theta_2^4 \sin\phi_1^4 = \frac{1}{6}\alpha_1\alpha_2|\overline{P_2P_4}| \sin\theta_2^4 \sin\phi_2^4,$$

we have

$$\frac{H_T}{h_T} = \frac{\alpha_1\alpha_2\alpha_3}{|T|} = \frac{6}{\sin\theta_2^4 \sin\phi_1^4} \le \frac{6}{\sin\theta_2^4 \sin\phi_2^4}. \tag{16}$$

From the definition of Type 2, we realize that $\theta_1^4 \le \theta_2^4 \le \theta_3^4$ on F_4, $\theta_2^3 \le \theta_1^3 \le \theta_4^3$ on F_3, and θ_2^1 is not the minimum angle of F_1. Thus, it follows from Lemma 14 that

$$\frac{\pi - \theta_{\max}}{2} \le \theta_2^4,\ \theta_1^3,\ \theta_2^1 \le \theta_{\max}, \quad \sin\theta_2^4,\ \sin\theta_1^3,\ \sin\theta_2^1 \ge C_1.$$

Additionally, we may apply Lemma 16 to θ_2^4 and F_4, and find that either $\psi^{1,4} \ge \delta$ or $\psi^{3,4} \ge \delta$, where $\delta = \delta(\theta_{\max})$ is defined by (15).

Suppose that $\psi^{3,4} \ge \delta$. Using the same argument as in Section 4.1, we have

$$\sin\phi_1^4 = \sin\theta_1^3 \sin\psi^{3,4} \ge C_0 \sin\theta_1^3 \ge C_0C_1.$$

Next, suppose that $\psi^{1,4} \ge \delta$. We have

$$\sin\phi_2^4 = \sin\theta_2^1 \sin\psi^{1,4} \ge C_0 \sin\theta_2^1 \ge C_0C_1.$$

Combining these results with (16), we obtain

$$\frac{H_T}{h_T} \le \frac{6}{C_0C_1^2} =: D,$$

that is, (3) holds.

§ 4.4. Type 2: Proof of “(3) implies MAC”

Finally, we suppose that T is of Type 2 and

$$\frac{H_T}{h_T} = \frac{\alpha_1\alpha_2\alpha_3}{|T|} = \frac{6}{\sin\theta_2^4 \sin\phi_1^4} \le D, \quad \sin\theta_2^4 \sin\phi_1^4 \ge \frac{6}{D} =: M.$$

The proof is very similar to that described in Section 4.2. By Lemma 18 with $j = 4$ and $p = 1$, setting $\gamma(M) := \pi - \sin^{-1} M$, the MAC with $\gamma(M)$ is satisfied on F_2, F_3, F_4, and $\psi^{2,4}$, $\psi^{3,4} \le \gamma(M)$.

Because $|\overline{P_2P_4}| \le \alpha_1$, we have

$$|T| = \frac{1}{6}|\overline{P_2P_3}||\overline{P_2P_4}||\overline{P_1P_4}| \sin\theta_2^1 \sin\phi_4^1 \le \frac{1}{6}\alpha_1\alpha_2\alpha_3 \sin\theta_2^1 \sin\phi_4^1.$$

This yields

$$D \ge \frac{H_T}{h_T} \ge \frac{6}{\sin\theta_2^1 \sin\phi_4^1} \quad \text{and} \quad \sin\theta_2^1 \sin\phi_4^1 \ge \frac{6}{D} = M,$$

and, by Lemma 18 with $j = 1$ and $p = 4$, we find that the MAC with $\gamma(M)$ is satisfied on F_1, and $\psi^{1,2}$, $\psi^{1,3} \le \gamma(M)$.

The final thing to prove is the MAC for $\psi^{1,4}$ and $\psi^{2,3}$. By the cosine rule (13) with $j = 2$, we have

$$\cos\psi^{1,4} = \sin\psi^{1,3}\sin\psi^{4,3}\cos\theta_2^3 - \cos\psi^{1,3}\cos\psi^{4,3},$$
$$\cos\psi^{2,3} = \sin\psi^{2,4}\sin\psi^{3,4}\cos\theta_1^4 - \cos\psi^{2,4}\cos\psi^{3,4}.$$

By the definition of Type 2, θ_2^3 and θ_1^4 are the minimum angles of F_3 and F_4, respectively. Therefore, we have $\cos\theta_2^3$, $\cos\theta_1^4 \ge \frac{1}{3}$ and thus

$$\cos\psi^{1,4} > -\cos\psi^{1,3}\cos\psi^{3,4}, \qquad \cos\psi^{2,3} > -\cos\psi^{2,4}\cos\psi^{3,4}.$$

Because $\sin\psi^{1,3}$, $\sin\psi^{2,4}$, $\sin\psi^{3,4} > M$, we find that

$$\cos\psi^{1,4} > -\cos\psi^{1,3}\cos\psi^{3,4} \ge -\sqrt{1-\sin^2\psi^{1,3}}\sqrt{1-\sin^2\psi^{3,4}} > M^2 - 1,$$
$$\cos\psi^{2,3} > M^2 - 1.$$

Therefore, we conclude that $\psi^{1,4}$, $\psi^{2,3} < \cos^{-1}(M^2-1) < \pi$, and T satisfies the MAC with

$$\theta_{\max} := \max\left\{\gamma(M), \cos^{-1}(M^2-1)\right\}.$$

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Appendix: Proof of Theorem 11 The proof of Theorem 11 is very similar to that of [13, Theorem 13] and [15, Theorem 21]. First, refer to [15, Section 5] for the definition of difference quotients of one and two variable functions. Difference quotients of three variable functions is their simple extension.

For a positive integer k, X^k is the set of lattice points defined by

$$X^k := \left\{ \mathbf{x}_\gamma := \frac{\gamma}{k} \in \mathbb{R}^3 \;\middle|\; \gamma \in \mathbb{N}_0^3 \right\},$$

where $\gamma/k = (a_1/k, a_2/k, a_3/k)$ is understood as the coordinate of a point in $\mathbb{R}^3$. For $\mathbf{x}_\gamma \in X^k$ and a multi-index $\delta \in \mathbb{N}_0^3$, we define the correspondence Δ^δ between nodes by $\Delta^\delta \mathbf{x}_\gamma := \mathbf{x}_{\gamma+\delta} = (\gamma+\delta)/k$.

For two multi-indexes $\eta = (m_1, m_2, m_3)$, $\delta = (n_1, n_2, n_3)$, $\eta \le \delta$ means that $m_i \le n_i$ $(i = 1, 2, 3)$. Also, $\delta \cdot \eta$ and $\delta!$ are defined by $\delta \cdot \eta := \sum_{i=1}^3 m_i n_i$ and $\delta! := n_1! n_2! n_3!$, respectively. Suppose that, for $\gamma, \delta \in \mathbb{N}_0^3$, both $\mathbf{x}_\gamma$ and $\Delta^\delta \mathbf{x}_\gamma$ belong to $\mathbf{K}$. Then, we define the *difference quotients* for $f \in C^0(\mathbf{K})$ by

$$f^{|\delta|}[\mathbf{x}_\gamma, \Delta^\delta \mathbf{x}_\gamma] := k^{|\delta|} \sum_{\eta \le \delta} \frac{(-1)^{|\delta|-|\eta|}}{\eta!(\delta-\eta)!} f(\Delta^\eta \mathbf{x}_\gamma).$$

For example, we see that

$$\begin{aligned} f^4[\mathbf{x}_{(0,0,0)}, \Delta^{(2,1,1)} \mathbf{x}_{(0,0,0)}] = \frac{k^4}{2} \big(& f(\mathbf{x}_{(2,1,1)}) - 2f(\mathbf{x}_{(1,1,1)}) + f(\mathbf{x}_{(0,1,1)}) \\ & - f(\mathbf{x}_{(2,0,1)}) + 2f(\mathbf{x}_{(1,0,1)}) - f(\mathbf{x}_{(0,0,1)}) \\ & - f(\mathbf{x}_{(2,1,0)}) + 2f(\mathbf{x}_{(1,1,0)}) - f(\mathbf{x}_{(0,1,0)}) \\ & + f(\mathbf{x}_{(2,0,0)}) - 2f(\mathbf{x}_{(1,0,0)}) + f(\mathbf{x}_{(0,0,0)}) \big). \end{aligned}$$

As explained in [15, Section 5], a differential quotients is expressed concisely by an integral. For that purpose, we introduce the s-simplex

$$\mathbb{S}_s := \left\{(x_1, \cdots, x_s)^\top \in \mathbb{R}^s \mid x_i \geq 0,\ 0 \leq x_1 + \cdots + x_s \leq 1\right\},$$

and the integral of $g \in L^1(\mathbb{S}_s)$ on $\mathbb{S}_s$ is defined by

$$\int_{\mathbb{S}_s} g(w_1, \cdots, w_k) \mathrm{d}\mathbf{W_s} := \int_0^1 \int_0^{w_1} \cdots \int_0^{w_{s-1}} g(w_1, \cdots, w_s) \mathrm{d}w_s \cdots \mathrm{d}w_2 \mathrm{d}w_1,$$

where $\mathrm{d}\mathbf{W_s} := \mathrm{d}w_1 \cdots \mathrm{d}w_s$. Then, $f^s[\mathbf{x}_{(l,q)}, \Delta^{(0,s,0)}\mathbf{x}_{(l,q)}]$ becomes

$$f^s[\mathbf{x}_{(l,q,r)}, \Delta^{(0,s,0)}\mathbf{x}_{(l,q,r)}] = \int_{\mathbb{S}_s} \partial^{(0,s,0)} f\left(\frac{l}{k}, \frac{q}{k} + \frac{1}{k}(w_1 + \cdots + w_s), \frac{r}{k}\right) \mathrm{d}\mathbf{W_s}.$$

For a general multi-index (t, s, m), we can write

$$f^{t+s+m}[\mathbf{x}_{(l,q,r)}, \Delta^{(t,s,m)}\mathbf{x}_{(l,q,r)}] = \int_{\mathbb{S}_s} \int_{\mathbb{S}_t} \int_{\mathbb{S}_m} \partial^{(t,s,m)} f\left(\mathbf{Z_t}, \mathbf{W_s}, \mathbf{Y_m}\right) \mathrm{d}\mathbf{Z_t} \mathrm{d}\mathbf{W_s} \mathrm{d}\mathbf{Y_m},$$

$$\mathbf{Z_t} := \frac{l}{k} + \frac{1}{k}(z_1 + \cdots + z_t),\ \mathrm{d}\mathbf{Z_t} := \mathrm{d}z_1 \cdots \mathrm{d}z_t, \quad \mathbf{W_s} := \frac{q}{k} + \frac{1}{k}(w_1 + \cdots + w_s),$$

$$\mathbf{Y_m} := \frac{r}{k} + \frac{1}{k}(y_1 + \cdots + y_m), \quad \mathrm{d}\mathbf{Y_m} := \mathrm{d}y_1 \cdots \mathrm{d}y_m.$$

Let $\square_\gamma^\delta$ be the rectangular parallelepiped defined by $\mathbf{x}_\gamma$ and $\Delta^\delta \mathbf{x}_\gamma$ as the diagonal points. If $\delta = (t, s, 0)$ or $(0, s, 0)$, $\square_\gamma^\delta$ degenerates to a rectangle or a segment. For $v \in L^1(\widehat{K})$ and $\square_\gamma^\delta$ with $\gamma = (l, q, r)$, we denote the integral as

$$\int_{\square_\gamma^{(t,s,m)}} v := \int_{\mathbb{S}_s} \int_{\mathbb{S}_t} \int_{\mathbb{S}_m} v\left(\mathbf{Z_t}, \mathbf{W_s}, \mathbf{Y_m}\right) \mathrm{d}\mathbf{Z_t} \mathrm{d}\mathbf{W_s} \mathrm{d}\mathbf{Y_m}.$$

If $\square_\gamma^\delta$ degenerates to a rectangle or a segment, the integral is understood as an integral on the rectangle or on the segment. By this notation, the difference quotient $f^{|\delta|}[\mathbf{x}_\gamma, \Delta^\delta \mathbf{x}_\gamma]$ is written as

$$f^{|\delta|}[\mathbf{x}_\gamma, \Delta^\delta \mathbf{x}_\gamma] = \int_{\square_\gamma^\delta} \partial^\delta f.$$

Therefore, if $u \in \mathcal{T}_p^k(\mathbf{T})$, then we have

$$0 = u^{|\delta|}[\mathbf{x}_\gamma, \Delta^\delta \mathbf{x}_\gamma] = \int_{\square_\gamma^\delta} \partial^\delta u, \qquad \forall \square_\gamma^\delta \subset \mathbf{T}. \tag{17}$$

Let $S \subset \mathbf{T}$ be a segment. In the proof of Theorem 11, the continuity of the trace operator t defined as $t : W^{1,p}(\mathbf{T}) \ni v \mapsto v|_S \in L^1(S)$ is crucial. For two-dimensional case, the continuity of t is standard and is mentioned in many textbooks such as [5]. For three dimensional case, the situation becomes a bit more complicated. If the continuous inclusion $W^{k+1,p}(\mathbf{T}) \subset C^0(\mathbf{T})$ holds, the continuity of t is obvious. Even if this is not the case, we still have the following lemma. For the proof, see [1, Theorem 4.12], [8, Lemma 2.2], and [17, Theorem 2.1].

Lemma 19. *Let $S \subset \mathbf{T}$ be an arbitrary segment. Then, the following trace operators are well-defined and continuous:*

$$t : W^{1,p}(\mathbf{T}) \to L^p(S), \quad 2 < p < \infty, \qquad t : W^{2,p}(\mathbf{T}) \to L^p(S), \quad 1 \le p < \infty.$$

Let p be taken as (2). The set $\Xi_p^{\delta,k} \subset W^{k+1-|\delta|,p}(\mathbf{T})$ is then defined by

$$\Xi_p^{\delta,k} := \left\{ v \in W^{k+1-|\delta|,p}(\mathbf{T}) \;\middle|\; \int_{\Box_\gamma^\delta} v = 0, \quad \forall \Box_\gamma^\delta \subset \mathbf{T} \right\}.$$

Note that $u \in \mathcal{T}_p^k(\mathbf{T})$ implies $\partial^\delta u \in \Xi_p^{\delta,k}$ by (17).

Lemma 20. *We have $\Xi_p^{\delta,k} \cap \mathcal{P}_{k-|\delta|} = \{0\}$. That is, if $q \in \mathcal{P}_{k-|\delta|}$ belongs to $\Xi_p^{\delta,k}$, then $q = 0$.*

Proof. Note that $\dim \mathcal{P}_{k-|\delta|} = \#\{\Box_\gamma^\delta \subset \mathbf{T}\}$. For example, if $k = 4$ and $|\delta| = 3$, then $\dim \mathcal{P}_1 = 4$. This corresponds to the fact that, in $\mathbf{T}$, there are four cubes of size $1/4$ for $\delta = (1,1,1)$ and there are four rectangles of size $1/2 \times 1/4$ for $\delta = (1,2,0)$. All their vertices (corners) belong to $\Sigma^4(\mathbf{T})$ (see Figure 7). Now, suppose that $q \in \mathcal{P}_{k-|\delta|}$ satisfies $\int_{\Box_\gamma^\delta} q = 0$ for all $\Box_\gamma^\delta \subset \mathbf{T}$. These conditions are linearly independent and determine $q = 0$ uniquely (see Exercise below). $\Box$

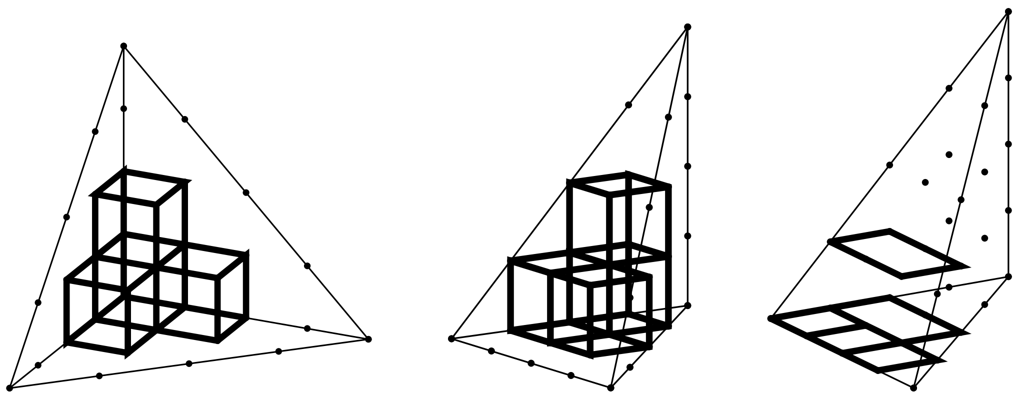

Figure 7. The four cubes and four rectangles in $\mathbf{T}$.

Exercise: Show that the condition "$\int_{\Box_\gamma^\delta} q = 0$ for all $\Box_\gamma^\delta \subset \mathbf{T}$" implies $q = 0$ for $q \in \mathcal{P}_{k-|\delta|}$. (Hint: (1) First, consider the case $d = 1$. For example, show the following: if a polynomial $p \in \mathcal{P}_k$ satisfies $\int_n^{n+1} p(x)\mathrm{d}x = 0$, $n = 0, \cdots, k$, then $p = 0$.)
(2) Reduce the proof of the case $d > 1$ to that of the case $d - 1$.

The constant $A_p^{\delta,k}$ is defined by

$$A_p^{\delta,k} := \sup_{v \in \Xi_p^{\delta,k}} \frac{|v|_{0,p,\mathbf{T}}}{|v|_{k+1-|\delta|,p,\mathbf{T}}}.$$

The following lemma is an extension of [3, Lemma 2.1].

Lemma 21. *Let p be such that $2 < p \le \infty$ if $k+1-|\delta| = 1$ or $1 \le p \le \infty$ if $k+1-|\delta| \ge 2$. We then have $A_p^{\delta,k} < \infty$.*

Proof. The proof is by contradiction. Suppose that $A_p^{\delta,k} = \infty$. Then there exists a sequence $\{w_k\}_{i=1}^\infty \subset \Xi_p^{\delta,k}$ such that $|w_n|_{0,p,\mathbf{T}} = 1$ and $\lim_{n\to\infty} |w_n|_{k+1-|\delta|,p,\mathbf{T}} = 0$. By the Bramble–Hilbert lemma [15, Theorem 14], there exists $\{q_n\} \subset \mathcal{P}_{k-|\delta|}$ such that

$$\|w_n + q_n\|_{k+1-|\delta|,p,\mathbf{T}} \le \inf_{q\in\mathcal{P}_{k-|\delta|}} \|w_n + q\|_{k+1-|\delta|,p,\mathbf{T}} + \frac{1}{n} \le C|w_n|_{k+1-|\delta|,p,\mathbf{T}} + \frac{1}{n}$$

and $\lim\limits_{n\to\infty} \|w_n + q_n\|_{k+1-|\delta|,p,\mathbf{T}} = 0$. Because $\{w_n\} \subset W^{k+1-|\delta|,p}(\mathbf{T})$ is bounded, $\{q_n\} \subset \mathcal{P}_{k-|\delta|}$ is bounded as well. Hence, there exists a subsequence $\{q_{n_i}\}$ such that q_{n_i} converges to $\bar{q} \in \mathcal{P}_{k-|\delta|}$ and $\lim_{n_i\to\infty} \|w_{n_i} + \bar{q}\|_{k+1-|\delta|,p,\mathbf{T}} = 0$. If $\Box_{lp}^\delta$ is not degenerate to a rectangle or a segment, we have

$$\left|\int_{\Box_{lp}^\delta} (w_{n_i} + \bar{q})\right| \le \int_{\Box_{lp}^\delta} |w_{n_i} + \bar{q}| \le C\|w_{n_i} + \bar{q}\|_{k+1-|\delta|,p,\mathbf{T}} \to 0 \quad \text{as } h \to 0. \tag{18}$$

If $\Box_{lp}^\delta$ is degenerate to a rectangle or a segment, (18) holds as well by Lemma 19. Because $\int_{\Box_{lp}^\delta} w_{n_i} = 0$ by the definition, we have

$$0 = \lim_{n_i\to\infty} \int_{\Box_{lp}^\delta} (w_{n_i} + \bar{q}) = \int_{\Box_{lp}^\delta} \bar{q}, \qquad \forall \Box_{lp}^\delta \subset \mathbf{T}.$$

Therefore, it follows from Lemma 19 that $\bar{q} = 0$. This implies that

$$0 = \lim_{n_i\to\infty} \|w_{n_i}\|_{k+1-|\delta|,p,\mathbf{T}} \ge \lim_{n_i\to\infty} |w_{n_i}|_{0,p,\mathbf{T}} = 1,$$

which is a contradiction. $\Box$

Define the linear transformation by, for $(x,y,z)^\top \in \mathbb{R}^3$,

$$(x^*, y^*, z^*)^\top = D_{\alpha_1\alpha_2\alpha_3}(x,y,z)^\top = (\alpha_1 x, \alpha_2 y, \alpha_3 z)^\top, \qquad \alpha_i > 0,\ i = 1,2,3,$$

which the diagonal matrix $D_{\alpha_1\alpha_2\alpha_3}$ is defined by (8). This linear transformation squeezes the reference tetrahedron $\mathbf{T}$ perpendicularly to $T_{\alpha_1\alpha_2\alpha_3} = D_{\alpha_1\alpha_2\alpha_3}\mathbf{T}$. Take an arbitrary $v \in \mathcal{T}_p^k(T_{\alpha_1\alpha_2\alpha_3})$ and define $u \in \mathcal{T}_p^k(\mathbf{T})$ by $u(x,y,z) := v(D_{\alpha_1\alpha_2\alpha_3}(x,y,z)^\top)$. Let p be taken as (2) with $m = |\delta|$. To make formula concise, we introduce the following notation. For a multi-index $\gamma = (a,b,c) \in \mathbb{N}_0^3$ and a real $t \ne 0$, and $(\alpha) := (\alpha_1, \alpha_2, \alpha_3)$, $(\alpha)^{\gamma t} := \alpha_1^{at}\alpha_2^{bt}\alpha_3^{ct}$. Because $u \in \mathcal{T}_p^k(\mathbf{T})$ and $\partial^\delta u \in \Xi_p^{\delta,k}$, we may apply Lemma 21 as follows. For p, $1 \le p < \infty$, we have

$$\frac{|v|_{m,p,T_{\alpha_1\alpha_2\alpha_3}}^p}{|v|_{k+1,p,T_{\alpha_1\alpha_2\alpha_3}}^p} = \frac{\sum_{|\gamma|=m} \frac{m!}{\gamma!}(\alpha)^{-\gamma p}\, |\partial^\gamma u|_{0,p,\mathbf{T}}^p}{\sum_{|\delta|=k+1} \frac{(k+1)!}{\delta!}(\alpha)^{-\delta p}\, |\partial^\delta u|_{0,p,\mathbf{T}}^p}$$

$$
\begin{aligned}
&= \frac{\sum_{|\gamma|=m} \frac{m!}{\gamma!} (\alpha)^{-\gamma p} \, |\partial^\gamma u|_{0,p,\mathbf{T}}^p}{\sum_{|\gamma|=m} \frac{m!}{\gamma!} (\alpha)^{-\gamma p} \left(\sum_{|\eta|=k+1-m} \frac{(k+1-m)!}{\eta! (\alpha)^{\eta p}} \, |\partial^\eta (\partial^\gamma u)|_{0,p,\mathbf{T}}^p \right)} \\
&\le \frac{(\max_{i=1,2,3} \alpha_i)^{(k+1-m)p} \sum_{|\gamma|=m} \frac{m!}{\gamma!} (\alpha)^{-\gamma p} \, |\partial^\gamma u|_{0,p,\mathbf{T}}^p}{\sum_{|\gamma|=m} \frac{m!}{\gamma!} (\alpha)^{-\gamma p} \left(\sum_{|\eta|=k+1-m} \frac{(k+1-m)!}{\eta!} \, |\partial^\eta (\partial^\gamma u)|_{0,p,\mathbf{T}}^p \right)} \\
&= \frac{(\max_{i=1,2,3} \alpha_i)^{(k+1-m)p} \sum_{|\gamma|=m} \frac{m!}{\gamma!} (\alpha)^{-\gamma p} \, |\partial^\gamma u|_{0,p,\mathbf{T}}^p}{\sum_{|\gamma|=m} \frac{m!}{\gamma!} (\alpha)^{-\gamma p} \, |\partial^\gamma u|_{k+1-m,p,\mathbf{T}}^p} \\
&\le \frac{(\max_{i=1,2,3} \alpha_i)^{(k+1-m)p} \sum_{|\gamma|=m} \frac{m!}{\gamma!} (\alpha)^{-\gamma p} |\partial^\gamma u|_{0,p,\mathbf{T}}^p}{\sum_{|\gamma|=m} \frac{m!}{\gamma!} (\alpha)^{-\gamma p} \left(A_p^{\gamma,k} \right)^{-1} |\partial^\gamma u|_{0,p,\mathbf{T}}^p} \\
&\le C_{k,m,p}^p \left(\max_{i=1,2,3} \alpha_i \right)^{(k+1-m)p}, \qquad (19)
\end{aligned}
$$

where $C_{k,m,,p} := \max_{|\gamma|=m} A_p^{\gamma,k}$. Here, we use the equality

$$
\frac{(k+1)!}{\delta!} = \sum_{\substack{\gamma+\eta=\delta \\ |\gamma|=m, |\eta|=k+1-m}} \frac{m!}{\gamma!} \frac{(k+1-m)!}{\eta!}.
$$

Hence, Theorem 11 is proved for this case. The proof of the case $p=\infty$ may be done in a similar manner. □

Exercise: (1) Check the above proof in detail. For example, confirm that, if $k=m=1$, (19) can be written as

$$
\begin{aligned}
\frac{|v|_{1,p,T_{\alpha_1\alpha_2\alpha_3}}^p}{|v|_{2,p,T_{\alpha_1\alpha_2\alpha_3}}^p} &= \frac{\sum_{|\gamma|=1} \frac{1}{\gamma!} (\alpha)^{-\gamma p} \, |\partial^\gamma u|_{0,p,\mathbf{T}}^p}{\sum_{|\delta|=2} \frac{2!}{\delta!} (\alpha)^{-\delta p} \, |\partial^\delta u|_{0,p,\mathbf{T}}^p} \\
&= \frac{\frac{1}{\alpha_1^p} |\partial_x u|_0^p + \frac{1}{\alpha_2^p} |\partial_y u|_0^p + \frac{1}{\alpha_3^p} |\partial_z u|_0^p}{\frac{1}{\alpha_1^{2p}} |\partial_{xx} u|_0^p + \frac{1}{\alpha_2^{2p}} |\partial_{yy} u|_0^p + \frac{1}{\alpha_3^{2p}} |\partial_{zz} u|_0^p + \frac{2}{\alpha_1^p \alpha_2^p} |\partial_{xy} u|_0^p + \frac{2}{\alpha_2^p \alpha_3^p} |\partial_{yz} u|_0^p + \frac{2}{\alpha_3^p \alpha_1^p} |\partial_{zx} u|_0^p} \\
&= \frac{\frac{1}{\alpha_1^p} |\partial_x u|_0^p + \frac{1}{\alpha_2^p} |\partial_y u|_0^p + \frac{1}{\alpha_3^p} |\partial_z u|_0^p}{\frac{1}{\alpha_1^p} X + \frac{1}{\alpha_2^p} Y + \frac{1}{\alpha_3^p} Z} \\
&\quad \Bigl(X := \tfrac{1}{\alpha_1^p} |\partial_{xx} u|_0^p + \tfrac{1}{\alpha_2^p} |\partial_{xy} u|_0^p + \tfrac{1}{\alpha_3^p} |\partial_{xz} u|_0^p, \quad Y := \tfrac{1}{\alpha_1^p} |\partial_{xy} u|_0^p + \tfrac{1}{\alpha_2^p} |\partial_{yy} u|_0^p + \tfrac{1}{\alpha_3^p} |\partial_{yz} u|_0^p, \\
&\qquad Z := \tfrac{1}{\alpha_1^p} |\partial_{zx} u|_0^p + \tfrac{1}{\alpha_2^p} |\partial_{zy} u|_0^p + \tfrac{1}{\alpha_3^p} |\partial_{zz} u|_0^p \Bigr) \\
&\le \frac{(\max_{i=1,2,3} \alpha_i)^p \left(\frac{1}{\alpha_1^p} |\partial_x u|_0^p + \frac{1}{\alpha_2^p} |\partial_y u|_0^p + \frac{1}{\alpha_3^p} |\partial_z u|_0^p \right)}{\frac{1}{\alpha_1^p} |\partial_x u|_1^p + \frac{1}{\alpha_2^p} |\partial_y u|_1^p + \frac{1}{\alpha_3^p} |\partial_z u|_1^p} \\
&\quad \Bigl(X \ge M |\partial_x u|_1^p, \quad Y \ge M |\partial_y u|_1^p, \quad Z \ge M |\partial_z u|_1^p, \quad M := (\max_{i=1,2,3} \alpha_i)^{-p} \Bigr)
\end{aligned}
$$

$$\leq \frac{(\max_{i=1,2,3} \alpha_i)^p \frac{(A_p^{(1,0,0),1})^p}{\alpha_1^p} |\partial_x u|_1^p + \frac{(A_p^{(0,1,0),1})^p}{\alpha_2^p} |\partial_y u|_1^p + \frac{(A_p^{(0,0,1),1})^p}{\alpha_3^p} |\partial_z u|_1^p}{\frac{1}{\alpha_1^p} |\partial_x u|_1^p + \frac{1}{\alpha_2^p} |\partial_y u|_1^p + \frac{1}{\alpha_3^p} |\partial_z u|_1^p}$$

$$\leq C_{1,1,p}^p \left(\max_{i=1,2,3} \alpha_i \right)^p, \qquad C_{1,1,p} := \max\{A_p^{(1,0,0),1}, A_p^{(0,1,0),1}, A_p^{(0,0,1),1}\}.$$

(2) Prove Theorem 11 for the case $p = \infty$.